A Two Population Model for the Stock Market Problem[1]

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Abstract: The development of the last year disaster in the Stock Markets all over the world gave rise to reconsidering the previous models used. It is clear that, even in an organized international or national context, large fluctuations and sudden losses may occur. This paper explores a two populations' model. The populations are conflicting into the same environment (a Stock Market) by following the main rules present, that is mutual interaction between adopters, potential adopters, word-of-mouth communication and of course by taking into consideration the innovation diffusion process. The proposed model has special futures expressed by third order terms providing characteristic stationary points.
Keywords: Chaotic modeling, The stock-market problem, Stock-Market, Innovation diffusion modeling, Lotka-Volterra, Simulation, Chaotic simulation.

1. Introduction

Several attempts to model the stock-market problem can be found in the literature. Between these models the Lotka-Volterra modeling approach is of considerable interest. The model can be used if we assume two interacting populations and can be generalized to more than two. The case of two interacting populations is explored by using two coupled differential equations and the results found give rise to a limit cycle and the corresponding oscillating graphs over time, Skiadas, 2009 [7]. However, the Lotka-Volterra system of differential equations, a non-linear system of the second order, fails to explain the sudden growth and decline resulting into the stock-market environment and even more the stability to high or low values (capital gains or losses). In view of the last year losses in the stock markets globally it would be very important to reconsider the classical Lotka-Volterra theory by introducing into the corresponding equations terms having to do with the diffusion aspect of the communication process. The proposed model has special futures expressed by third order terms providing characteristic stationary points. It should be noted that a seminal paper was published by Harold Hotteling [1] during March 1929 few months before the big crash in the New York stock exchange. Hotelling, in this paper explored the "*stability in competition*" problem. The analysis

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[1] Skiadas, C. H., A Two Population Model for the Stock Market Problem, in *Chaotic Systems: Theory and Applications*, C. H. Skiadas and I. Dimotikalis, Eds, World Scientific, 302-308, 2010.

proposed was based in a simplification of the problem. Only two markets where competing; however, the results where extremely important and clarified the situation. In this paper we accept the same methodology supposing that there is a stock market with two interacting populations and we explore the arising case. That is new is the introduction of the theories of the last decades on the adoption-diffusion of innovations in developing the equations of the proposed model.

2. The Model and Simulations

To model the specific situation we take into account that two populations x and y are present into the stock market and interact each other. Even more to simplify the case we suppose that the variables x_t and y_t stand for the number of players or for the number of transactions or for the values of the stocks belonging to each part of the players at a specific time period t. The aim is to explore the behavior of the two populations during time and especially in the limits.

A model including the main characteristics of two interrelating or even conflicting populations into a stock market can be expressed by the following set of differential equations:

$$\dot{x} = a_1 y + a_2 xy + a_3 x(l - y) + a_4 yx(k - x)$$
$$\dot{y} = b_1 x + b_2 xy + b_3 y(k - x) + b_4 xy(l - y)\ ,$$

where a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are parameters expressing the mutual interaction of the populations x and y. The parameters k and l express the upper limit of the populations x and y respectively.

The first term to the right stands for the flows from the one part to the other whereas the second term expresses the mutual interaction between the active parts of the populations x and y. If we retain the first two terms to the right the proposed model is of the Lotka-Volterra type.

The non active parts of the populations are $(k – x)$ for x and $(l - y)$ for the population y. The interaction of these parts with the active populations y and x is expressed with the third terms to the right of the above differential equations (see [2, 3, 4, 5, 6]).

The fourth terms to the right include the terms $x(k - x)$ and $y(l - y)$ multiplied by y and x respectively. These terms account to the rates of adoption-diffusion that is extremely important in order to express the word-of-mouth communication between adopters (the active part of the players) and potential adopters (the non-active part of the players).

Looking back to the set of the two differential equations above we see that we have two third degree equations for x and y. In the following we check the influence of the third order term to the stock-market model. To this end the above model is simplified to the following form:

$$\dot{x} = -ay + cyx^2$$
$$\dot{y} = bx - cxy^2 \, ,$$

Where *a*, *b* and *c* are parameters of interaction. Especially the parameter *c* is selected to be the same in both equations expressing the coupling of both populations *x* and *y*.
The set of the last equations takes the form:

$$\dot{x} = -ay + cyx^2 = -cy\sqrt{a/c}\left(\sqrt{a/c} - x\right) - cxy\left(\sqrt{a/c} - x\right)$$
$$\dot{y} = bx - cxy^2 = cx\sqrt{b/c}\left(\sqrt{b/c} - y\right) + cxy\left(\sqrt{b/c} - y\right),$$

The last equations include in the right part side two basic parts of the interaction and the diffusion process where $x = \sqrt{a/c}$ and $y = \sqrt{b/c}$ are the upper limit or the maximum for *x* and *y* respectively.
There is a fixed point at (0, 0) and four other characteristic points at

$$(x = \sqrt{a/c} \, , y = \sqrt{b/c} \,), \;\; (x = \sqrt{a/c} \, , y = -\sqrt{b/c} \,)$$
$$(x = -\sqrt{a/c} \, , y = \sqrt{b/c} \,), (x = -\sqrt{a/c} \, , y = -\sqrt{b/c} \,)$$

The last differential equations give the following differential equation for *x* and *y*:

$$\frac{dy}{dx} = \frac{bx - cxy^2}{-ay + cyx^2}$$

The solution is

$$(y^2 - b/c)(x^2 - a/c) = h$$

Where *h* is the integration constant.
The next Figure illustrates characteristic graphs of the last equation. The four characteristic points define a rectangle. Into this space are drawn few of the trajectories of the process. The trajectories outside of the rectangle diverge to infinity.

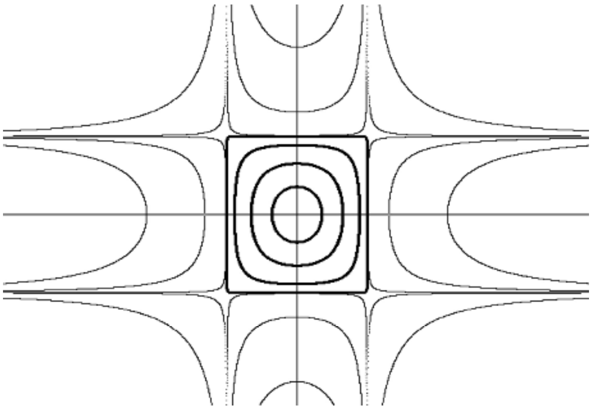

Figure 1. Characteristic Trajectories for various *h*.

As it is presented in Figure 1 there appear to be positive and negative values for the stock-market process. To be realistic we will move the main part, that is the rectangle space, inside the first quarter (the positive space) of the Cartesian Coordinates. This is achieved by introducing the transformation $x^{*}=x-e_1$ and $y^{*}=y-e_2$.
The resulting set of differential equations is:

$$\dot{x} = -a(y-e_2) + c(y-e_2)(x-e_1)^2$$
$$\dot{y} = b(x-e_1) - c(x-e_1)(y-e_2)^2,$$

Where x stands for x^{*} and y for y^{*}.
We can also find the corresponding difference equation form for this system by observing that

$$\dot{x} = \frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = \frac{x_{t+1} - x_t}{(t+1)-t} = x_{t+1} - x_t$$
$$\dot{y} = \frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = \frac{y_{t+1} - y_t}{(t+1)-t} = y_{t+1} - y_t$$

The resulting difference equations set is:

$$x_{t+1} = x_t - a(y_t - e_2) + c(y_t - e_2)(x_t - e_1)^2$$
$$y_{t+1} = y_t + b(x_t - e_1) - c(x_t - e_1)(y_t - e_2)^2,$$

An illustration of this difference equations' model appear in Figure 2. There is an unstable fixed point located at (e_1, e_2) and the four fixed points at the corners of the rectangle. The process can stabilized at any of the four points depending on the starting values or on values of the parameters. That is extremely important is that we may have four specific cases when after some time the process is stabilized in which:

1. ($x = e_1 + \sqrt{a/c}$, $y = e_2 + \sqrt{b/c}$); both gains
2. ($x = e_1 + \sqrt{a/c}$, $y = e_2 - \sqrt{b/c}$); *x* gains, *y* losses
3. ($x = e_1 - \sqrt{a/c}$, $y = e_2 + \sqrt{b/c}$); *x* losses, *y* gains
4. ($x = e_1 - \sqrt{a/c}$, $y = e_2 - \sqrt{b/c}$); both losses

In Figure 2 the third case appears by means that *y* gains and *x* losses. The upper part of the figure illustrates the characteristic rectangle whereas the time varying development of *x* and *y* is presented. The original sinusoidal oscillations give rise to step like oscillations and finally to a stable process where *x* retain losses and *y* keep their gains. The process has stopped at the upper left hand side of the characteristic rectangle indicated by a small circle (see figure 2). The starting point was inside the characteristic rectangle and the movement was counter-clockwise.

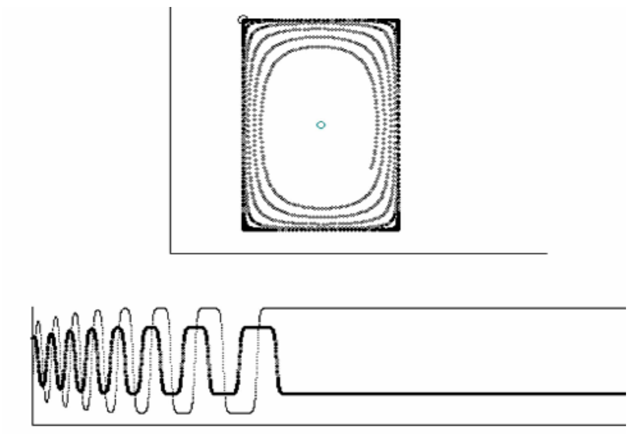

Figure 2. The Difference Equations model (Case 3)

For a simpler presentation of the findings we use the previous difference equation without the linear movement expressed by e_1 and e_2 by setting e_1=0 and e_2=0. The resulting form is

$$x_{t+1} = x_t - ay_t + cy_t x_t^2$$
$$y_{t+1} = y_t + bx_t - cx_t y_t^2,$$

The Jacobian of this map is

$$J(x,y)=\begin{vmatrix}1+2cxy & -a+cx^2\\ b-cy^2 & 1-2cxy\end{vmatrix}=1+ab-3c^2x^2y^2-c(ay^2+bx^2)$$

The value of the Jacobian at the characteristic point (0, 0) is:
$J(0,0)=1+ab$. Assuming positive parameters a and b the point (0,0) is unstable.
The Jacobian at the other 4 characteristic points $\left(\pm\sqrt{a/c},\pm\sqrt{b/c}\right)$ is
$J(0,0)=1-4ab$, and for appropriate positive values of the parameters the 4 points are stable (attracting).
The resulting graphs are presented in the following four figures (Fig.3–Fig.6) where the four different cases of gains and losses appear for various starting values of x_0 and y_0. The parameters selected are a=0.06, b=0.03 and c=0.025. A small circle in the appropriate corner of the characteristic rectangle indicates the stopping case. In Figure 3 this circle is at the upper right hand corner, in Figure 4 is at the lower right part of the rectangle whereas, in Figure 5 the circle is at the upper left part and in Figure 6 the circle is located at the lower left hand side. In the later case both x and y is stabilized at the lower values of the process with losses.

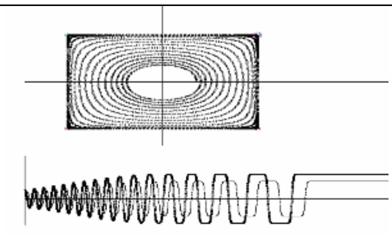

Fig. 3 (x_0=0.5, y_0=0)

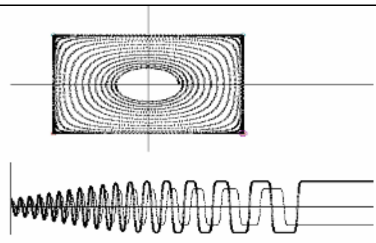

Fig. 4 (x_0=0.55, y_0=0)

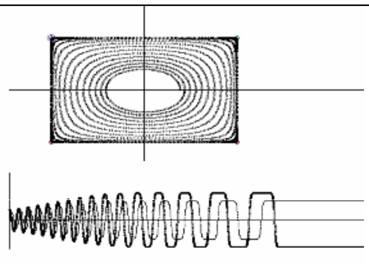

Fig. 5 (x_0=0.6, y_0=0)

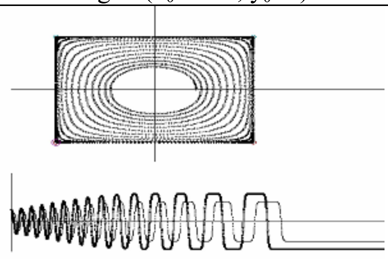

Fig. 6 (x_0=0.65, y_0=0)

Model Simulations

3. Conclusions

A model expressing two conflicting populations in the stock-market is developed. A general model is formulated and a simpler one is explored and simulated. The results support the well known process of fluctuations, oscillations and further sudden growth and decrease of gains-losses of the two conflicting populations. It was derived that it could arise that both would be stabilized to gains or losses or to stabilize the one in gains and the other to losses.

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