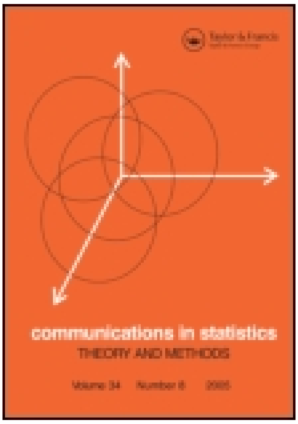

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The First Exit Time Theory Applied to Life Table Data: The Health State Function of a Population and Other Characteristics

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The First Exit Time Theory Applied to Life Table Data: The Health State Function of a Population and Other Characteristics

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In this article, we summarize the main parts of the first exit time theory developed in connection to the life table data and the resulting theoretical and applied issues. New tools arise from the development of this theory and especially the Health State Function and some important characteristics of this function.

We provide both simple and complex models and propose a methodology for reconstructing the health state function from the provided first exit time density distribution (for the appropriate computer program, see http://www.cmsim.net). In the simpler case, this theory is applied for the reconstruction of the so-called Inverse-Gaussian function.

Keywords Demographic analysis; First exit time probability density function; Health State Function; Hitting time model; Inverse Gaussian; Life Table Data; Stochastic modeling.

Mathematics Subject Classification 60H10; 62N05; 91D20; 97K06.

1. Introduction

1.1. *Extracting Information from Population-Mortality Data*

Over the centuries the systematic gathering of information for births-deaths from many countries and the related Census Bureaus or Statistical Agencies gave rise to theoretical and applied studies, both qualitative and quantitative in the areas of demography, probability and statistics, applied mathematics, and more recently, computer simulations.

Their main task was to extract as much information as possible from the existing data sets but also to propose and develop a framework for more effective

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gathering, storage and dissemination of population and mortality data. To this end, nowadays we have quite reliable data sets based on the theoretical background developed in the last centuries. Providing reliable data, systematically developed, is the basis of good qualitative and quantitative studies. For birth-death data related to a country the data sets are typically quite large, assuming that these data include in a statistically reliable way the main part of the information for the status of the population of the specific country over time and age group. These types of data sets include and describe quantitatively the course of the health state of the population including the births and deaths. At first glance, the data are simply sets for births and deaths. However, as these data systematically provide complete information for the status of the population by age group or by year of age, it is not so difficult to develop mathematical-statistical techniques to provide information for the health state or for the mortality of the population. From the beginning of the theoretical approaches, back to Graunt (1662), Halley (1693), and others four centuries ago, the task was to develop a theoretical framework under the title "Life Tables" facing mortality from the optimistic point of view. The level for the health state of the population emerged indirectly by estimating life expectancies and the life expectancy at birth. Halley recommended the use of a term related to the level of the health state called "vitality" to describe the gradual deterioration of the human health in the course of time. However, the main term that became accepted and used was the "force of mortality" per age. It was not evident how it would be possible to propose and estimate a similar term for the state of the human health or for the vitality of an organism. It is surprising because it is the health state that is the *direct dominating effect* during the life time, and not the inverse, the force of mortality.

The health state or the vitality of an organism can be estimated from the birth-death data sets. But how? Qualitatively it is a common sense that a good health state of a population will result in higher life expectancy. Quantitatively the answer should be included into the birth-death data and mainly into the death data sets per age or per year of age.

But how can one quantify and model the health state of a population by providing a health state function per age? The cause of the lack of developing a quantitative theory is mainly due to the fact that the health state of an individual is a stochastic process and the death is the end of this process when the health state drops below a limit, called a "barrier" in terms of the first exit time or hitting time stochastic theory. The theory of stochastic processes was developed only in the last century and a well-established theory for the first exit time processes was proposed only during the last decades. By contrast, the force of mortality was based on simple probabilistic arguments developed over the last four centuries.

Another point is that a well-established and developed theory for the force of mortality originated from Gompertz (1825) and the related theory based on his article is widely known and used in demography and actuarial studies and applications. Why should one then use a new term and a new theory based on the health state of a population and the related function? To be viable and have a wide spread and applicability, a new theory on the health state of a population needs to fulfill the following:

1. to be based on a sound theoretical background;
2. to include the previous theoretical achievements and especially the functions proposed for the force of mortality and the models proposed;
3. to include functions and methods improving the life expectancy estimates;

4. to provide new tools and especially tools related to the estimation of the health state of a population. These tools should include estimates for the "age of the maximum health state" and the "level" of this health state; and
5. to be supplemented with the necessary software for immediate application and testing.

For several years we have worked on exploring the health state process and providing the material in order to give a quite strong scientific tool useful to theoreticians and practitioners.

In this article, we present the main parts of the theory followed by direct applications to data sets.

While the literature regarding our method for the health state function is missing thus making it impossible to make comparisons with other health state functions proposed we have succeeded to propose a method to construct the health state of a population only from data, thus avoiding the calculation of the parameters of a model by fitting model to data. This is important for demographers, actuaries, and policy makers to calculate the health state of a particular population, to explore for changes over time and compare the health status between countries and geographic regions. A good fitting model as those we have proposed may provide additional information while it is useful for "smoothing" the data in cases with systematic fluctuations.

The next step is the use of the new methods and techniques to estimate characteristic parameters of a population system related to the healthy life years and the healthy life expectancy or to explore future trends of health and mortality.

The forecasting ability of the proposed health state function should be tested further in future studies with the results provided by other methods and especially with the modeling technique proposed by Lee and Carter (1992) and further developed and applied by Brouhns et al. (2002) and Haberman and Renshaw (2008). Further applications can be done in the directions given in Pitacco et al. (2009) and Pitacco and Olivieri (2009), focusing on actuarial and social and economic health systems.

1.2. *Theoretical Background*

It is expected that a probability density function expressing the death process could arise by using the stochastic theory related to the first exit time, via estimating the probability density function concerning the first time for a stochastic process to cross a barrier. The related theory originates from the pioneering work of Schrödinger and Smoluchowsky in 1915. For the case of the death density function the theoretical attempts where based on "vitality" a term originated from Halley (1693), Gompertz (1825), and few decades ago, Strehler and Mildvan (1960). Death arises as a cause of the loss of vitality or health which can be considered as a stochastic process. Such a process can be modeled by a simple stochastic differential equation of the form

$$dS_t = \mu_t^* dt + \sigma_t dW_t, \tag{1}$$

where S_t is the health state or the vitality of the individual, μ_t^* is a function of the age t, and σ_t is the diffusion coefficient. In order to avoid confusion with the force of mortality denoted by μ in the actuarial science, we have set the μ^* for the different function expressing the loss of vitality or the rate of decrease of the health state.

If we assume that μ_t^* and σ_t are real functions the solution of (1) is immediate by integration:

$$S_t = \int_0^t \mu_s^* ds + \int_0^t \sigma_s dW_s. \tag{2}$$

We set

$$\mu_t^* = dH_t/dt, \tag{3}$$

where H_t is the health state function.

The main problem here is not to find the solution of (1) but the transition probability density function $p(t)$. From (1) we can pass to the associated Fokker-Planck partial differential equation:

$$\frac{\partial p(S_t, t)}{\partial t} = -\mu_t^* \frac{\partial p(S_t, t)}{\partial S_t} + \frac{\sigma_t^2}{2} \frac{\partial^2 p(S_t, t)}{\partial S_t^2}. \tag{4}$$

The solution is given in Janssen and Skiadas (1995) and is of the form

$$p(t) = \frac{1}{\left[2\pi \int_0^t \sigma_s^2 ds\right]^{1/2}} e^{-\frac{(H_t)^2}{2\int_0^t \sigma_s^2 ds}}, \tag{5}$$

which, for constant σ, takes the form

$$p(t) = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(H_t)^2}{2\sigma^2 t}}. \tag{6}$$

2. The First Exit Time Density Function

The method for finding a density function expressing the distribution of the first exit time of particles escaping from a boundary was developed by Schrödinger (1915) and Smoluchowsky (1915) in two papers published independently in the same journal issue. Later on, Siegert (1951) gave an interpretation closer to our modern notation whereas Jennen (1985), Lerche (1986), and Jennen and Lerche (1981) gave the most interesting first exit time density function form. For the simple case presented earlier in (6), the proposed form is:

$$g(t) = \frac{|a|}{t} p(a, t) = \frac{|a|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{a^2}{2\sigma^2 t}}, \tag{7}$$

where a is the distance from the barrier.

Jennen (1985) proposed a more general form for the case of curved boundary using a tangent approximation of the first exit time density. More recent works can be found in Wang and Pötzelberger (1997), Peskir (2002), and Zucca and Sacerdote (2009). The simplest approach in modeling mortality leads to the following form (earlier work can be found in Janssen and Skiadas, 1995; Skiadas, 2011a, b; Skiadas and Skiadas, 2007, 2010a, b; 2011a, b):

$$g(t) = \frac{|H_t - tH_t'|}{t} p(t) = \frac{|H_t - tH_t'|}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(H_t)^2}{2\sigma^2 t}}, \tag{8}$$

where H_t' is the derivative of H_t.

As we can see from the last form, the term $|H_t - tH_t'|$ accounts for the following formula arising from a Taylor series expansion

$$H_t - tH' = H_0 + \frac{t^2}{2}H'' + \ldots \tag{8a}$$

For the case of a smooth slowly varying health state function H_t the error is of the order of H''. For fitting the simpler approximation form arises with a cost of estimating the correct H_t:

$$g(t) = \frac{k}{t}p(t) = \frac{k}{\sigma\sqrt{2\pi t^3}}e^{-\frac{(H_t)^2}{2\sigma^2 t}}. \tag{9}$$

The equation form (9) is simpler than Eq. (8) and very easy to fit to data sets considering that many non-linear parameters have to be estimated. From this point of view (9) gave almost perfect fitting in many applications.

We arrived at the simpler form (9), very important for the proposed theory in order to be able to have a closed form solution for the health state function H_t by rearranging (9), with a cost of an error of estimating H_t. This error was observed as to be small when we had performed some thousand of stochastic simulations in Janssen and Skiadas (1995).

However, we can estimate the approximation term by introducing a function f_t in (9) as follows

$$g(t) = \frac{k}{t}p(t) = \frac{k}{\sigma\sqrt{2\pi t^3}}e^{-\frac{(H_t - f_t)^2}{2\sigma^2 t}} \tag{9a}$$

Then one task is to find the function f_t so that (9a) to be a good approximation of (8). In several cases it is better to find f_t by other methods as are the simulation techniques instead of approximations, because (8) is also an approximation and perhaps is not good enough for a large interval of t. Fortunately, a simple function for f_t of the form

$$f_t = at^{\beta} \tag{9b}$$

is convenient for the cases related to the health state function of a population, and mortality and population data used in our applications, as we will demonstrate in the simulations section. The other problem on how to find the parameter k is given in Sec. 3.

By estimating the parameters of H_t when fitting a good model to data we observed that the fitting error was almost negligible for data of countries like the U.S. where good methods of collecting data are used along with the large number of population and deaths, thus providing a very smooth distribution for $g(t)$. It comes out that the term $|H_t - tH_t'|$ which was replaced by the constant k if not a constant should be included into the only time varying function H_t as a part of this function (σ is also considered as constant in all the applications). The second step is to solve the inverse problem to find the error term or better the correction function by a stochastic simulation method. A complete presentation should be given in a specific paper. In the simulations section we give few examples and applications.

What remains is to find the form of the health state function H_t expressing the loss of vitality process utilizing that it must be a declining function at the end of the life course. A simple such form is expressed by (Skiadas and Skiadas 2010a, 2010b):

$$H_t = l - (bt)^c, \tag{10}$$

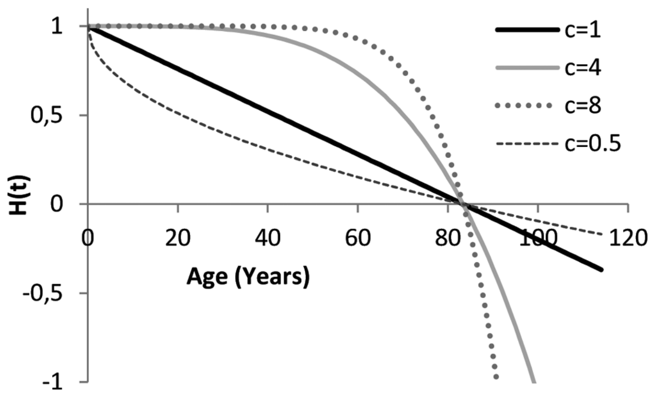

Figure 1. Simple health state models.

where l, b and c are parameters. When $c = 1$, the resulting probability density function is the so-called Inverse Gaussian

$$g(t) = \frac{k}{\sigma\sqrt{2\pi t^3}} e^{-\frac{(l-bt)^2}{2\sigma^2 t}}. \tag{11}$$

The effect of the health state decline modeling is illustrated in Fig. 1. The linear approach ($c = 1$) is possible only for a very simple system or organism. The death process is expressed by a gradual deterioration of the system without repairing mechanisms. As the organism or the system becomes more complicated and repair mechanisms are present, the parameter c is becomes larger, thus modeling an organism which remains in a good condition for a large period of the life span. These cases are expressed by $c = 4$ and $c = 8$. The fast decreasing period is due to the code governing the health state. The case with $c = 0.5$ accounts for a system not well operating. Figure 2 illustrates the first exit time densities for the cases of Fig. 1.

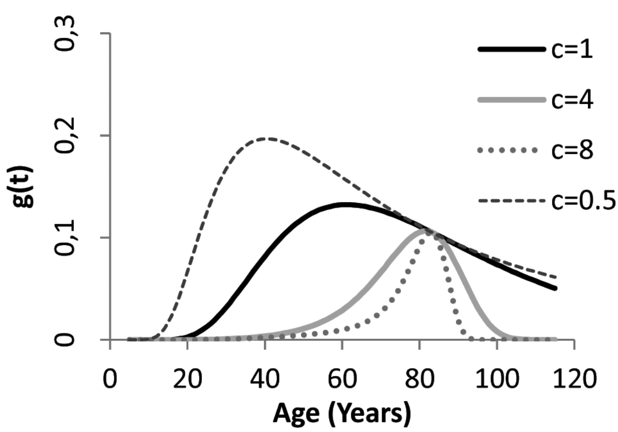

Figure 2. First exit time densities for models (9) and (10) for $k = 10$, $l = 1$, $b = 0.012$, and $\sigma = 0.05$.

3. A Special Property of the Health State Function $H(t)$

As we already introduced the health state as the opposite of mortality and the health state function, as the opposite of mortality function, we should find a simple but quite effective method to estimate $H(t)$ from Life Tables or from Mortality Data. From the first introduction of life tables it was evident that the optimistic term "life" was preferred than the pessimistic "mortality". Gompertz (1825) suggested the use of "life contingencies" in his seminal paper "On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies". Halley (1693) estimated mortality in his paper, "An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives". The terms Life Table and Life Table Data are more frequently used especially from actuaries as such terms emphasize the optimistic view of the life course.

As the death data from a country include, on average, a large part of the information related to the health state of the population, it is surprising that no-quantitative works are present related to the health state of the population during the last centuries. We understand that the health state data are included into the mortality data and the main task is to find a method to extract this information from the mortality data or from the life table data. It is clear that both population data and mortality data should contribute as is already the case when constructing life tables.

The theory of the proposed health state function is relatively new. Special features are expected to arise by continuing study and research. More work is needed especially on finding the form of the unknown function $H(t)$ and the properties of this function. The simpler idea in exploring the form of $H(t)$ is to reconstruct it from the existing data sets of a population. The data usually refer to a country. To reconstruct $H(t)$ we need to reformulate the probability density function $g(t)$ expressed by Eq. (9) in terms of $H(t)$. Then, as the data for $g(t)$ are obtained from the bureau of the census or official databases it will be possible to estimate $H(t)$. Turning back to the general equation proposed in Eq. (9), we see that by estimating the parameters of a model with any health state function $H(t)$ the diffusion coefficient σ cannot be estimated explicitly and it will be an internal part of these parameters. So that we transform Eq. (9) in the following form:

$$g(t) = \frac{\left(k^*/(\sigma\sqrt{2\pi})\right)}{\sqrt{t^3}} e^{-\frac{(H_t^*/\sigma)^2}{2t}}. \tag{12}$$

By setting

$$k = k^*/\left(\sigma\sqrt{2\pi}\right), \tag{13}$$

and

$$H_t = H_t^*/\sigma, \tag{14}$$

we arrive to the following simpler form which is more convenient for the applications

$$g(t) = \frac{k}{\sqrt{t^3}} e^{-\frac{(H_t)^2}{2t}}. \tag{15}$$

We now turn our attention to producing an estimate of the form of the unknown health state function $H(t)$ by rearranging the previous equation and expressing $H(t)$ as a function of $g(t)$. The resulting function is of the form

$$H_t^{=} \pm \left| \left(-2t \ln \frac{g(t)\sqrt{t^3}}{k} \right)^{1/2} \right|. \tag{16}$$

To assure a positive sign in the between brackets term on the right-hand side of the last formula the following relation must hold:

$$k \geq g(t)\sqrt{t^3}. \tag{17}$$

Thus, we can immediately have an illustration of the form of the function $H(t)$ by introducing the values for the deaths $g(t)$ per age t from any annual data sets from the human mortality databases in the above formula after estimating the parameter k.

Application to the mortality data of U.S. for 2000, provided by the Human Mortality Database, gave $k = 31.10$ at 88 years of age for females (Fig. 3) and $k = 25.49$ at 83 years of age for males (Fig. 4).

The very interesting finding is that along with the characteristic value of $k(T)$ we have an estimate for the age of the maximum death rate T from the relation:

$$k(t) = g(t)\sqrt{t^3}. \tag{18}$$

The estimation of the parameter k is an easy task if we observe that the function $H(t)$ should be continuous as is the function $g(t)$, it should decrease continuously in the older ages and it should be zero (or very close to zero) at the age of the maximum death rate. As illustrated in Fig. 3, there is only one curve fulfilling the previous arguments when $k = 31.10$ (U.S. 2000, females) whereas the cases with $k < 31.10$ violate continuity (see the curves for $k = 5$) and the cases with $k > 31.10$ show a gradual increase in the old ages (see the curves for $k = 60$). In the same figure we have indicated the form which the unknown function $H(t)$ should have

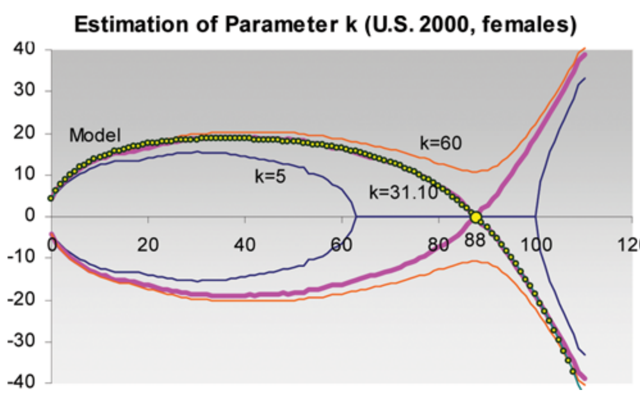

Figure 3. The parameter $k(t)$ for females in the U.S. (2000). Two cases not appropriate with $k = 5$ and $k = 60$ and the correct $k = 31.10$ at 88 years of age are presented.

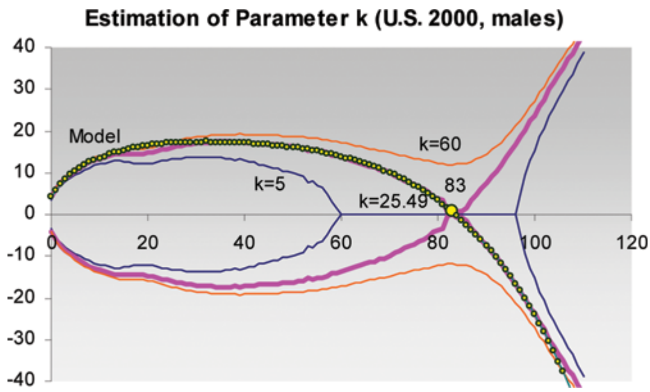

Figure 4. The parameter $k(t)$ for males in the U.S. (2000). Two cases not appropriate with $k = 5$ and $k = 60$ and the correct $k = 25.49$ at 83 years of age are presented.

(curve with small circles). Similar results are illustrated in the next Fig. 4 (U.S. 2000, males).

The Health State Functions for males and females for the same time period (2000) for the U.S. are illustrated in Fig. 5a. The $H(t)$ for females has higher values than for males and the same holds for the age of zero $H(t)$. In both cases, the Maximum Health State is found to be between 30 and 45 years of age, a quite reasonable finding. An important estimator is found by calculating the area between the health state curve and the horizontal axis (see Fig. 5a). This estimator accounts for the total health state (THS) of a population (Skiadas and Skiadas, 2013). Changes of this estimator provide important information for the health state condition of a population especially when are compared with the life expectancy at birth (LEB). The results of an application in Sweden, females (1751–2007) for LEB vs. THS are illustrated in Fig. 5b. As it was expected the life expectancy at birth is improved as the health state of the population is getting higher. However, the LEB development is slower than the THS.

While the Health State Function can be reconstructed from data without using a model we have tested a model proposed by Janssen and Skiadas (1995) of the form (15) where the health state function H_t is

$$H_t = a_1 + at^4 - b\sqrt{t} + lt^2 - ct^3. \tag{19}$$

This model is developed in order to include the infant mortality (a_1, a, b, l and c are parameters). The results from fitting the model to data (dotted curves in Figs. 3 and 4) follow quite well the estimates with the reconstruction method.

The proposed technique is very general and can be applied for simpler forms of health state functions as given by Eq. (10). Two forms for $c = 1$ and $c = 4$ are illustrated in the following figures. In both cases, the data used for the graphs in Figs. 1 and 2 are introduced. In the first case the Inverse Gaussian ($c = 1$) is reconstructed. First we formulate the function $k(t)$ presented in Fig. 6. This graph provides the maximum value for k and the age year with zero health state $H = 0$. The estimated value for the maximum is $k = 79.79$. This is the value for k which provides the correct shape for the $H(t)$ of the Inverse Gaussian. This is expressed by the straight line in Fig. 7. All other values for k fail to give the correct form of

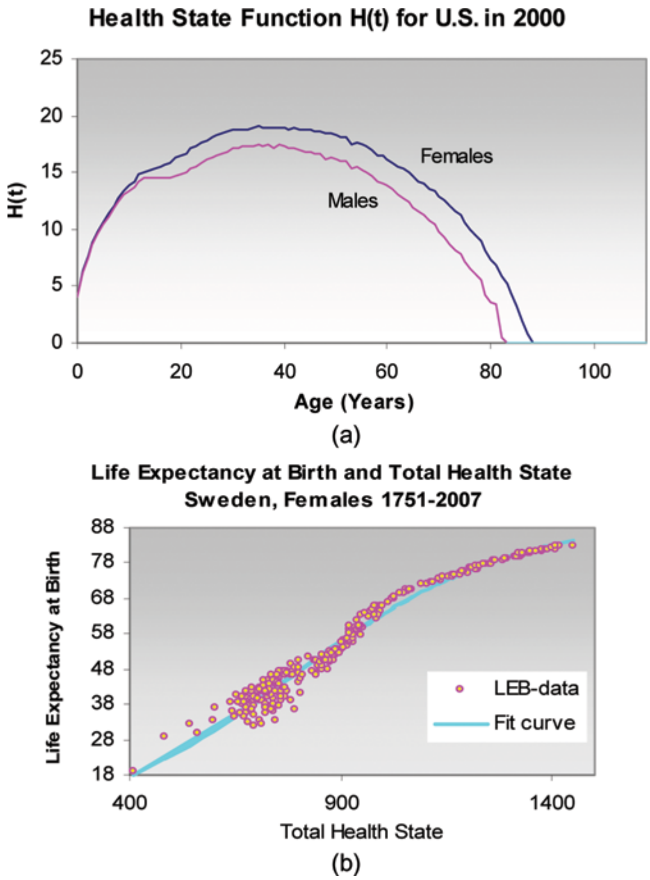

Figure 5. a. Health state function for males and females in the U.S.; b. Life expectancy at birth versus total health state in Sweden.

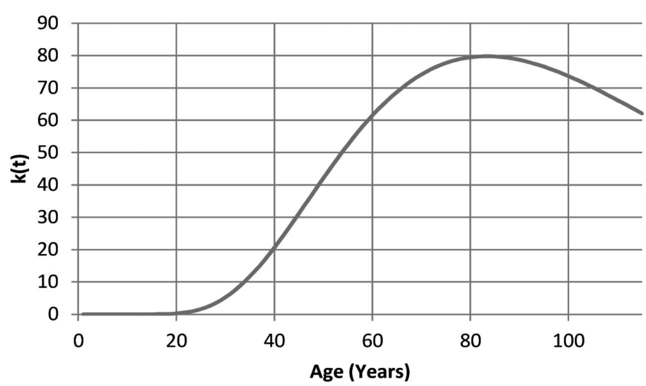

Figure 6. Estimation of $k(t)$ for the Inverse Gaussian.

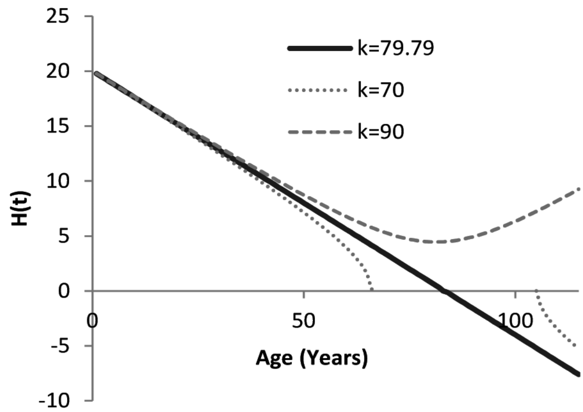

Figure 7. Reconstruction of $H(t)$ for the Inverse Gaussian ($c = 1$).

$H(t)$. Two cases with $k = 70$ and $k = 90$ are presented in the same Fig. 7. In the first case with $k = 70$, $H(t)$ splits in two non-connected curves whereas in the second case $k = 90$ leads to a declining first and then increasing curve.

It must be noted that the estimated values for k and $H(t)$ come from the relations (16) and (17), respectively.

Figures 8 and 9 illustrate the results for the model proposed in Skiadas and Skiadas 2010 for the special case with $c = 4$. Figure 8 provides the form of $k(t)$ and Fig. 9 shows the characteristic shape for the health state function $H(t)$. As in the previous case the maximum is $k = 79.79$. Two special cases with $k = 30$ and $k = 110$ are rejected, as well as all other values for k except $k = 79.79$. The case with $k = 30$ violates continuity and the other case with $k = 110$ provides a growing part in the end.

4. First Exit Time Stochastic Simulations

The main issue when doing stochastic simulations is that it is difficult or impossible in some cases to solve the inverse problem that is first to find the probability density

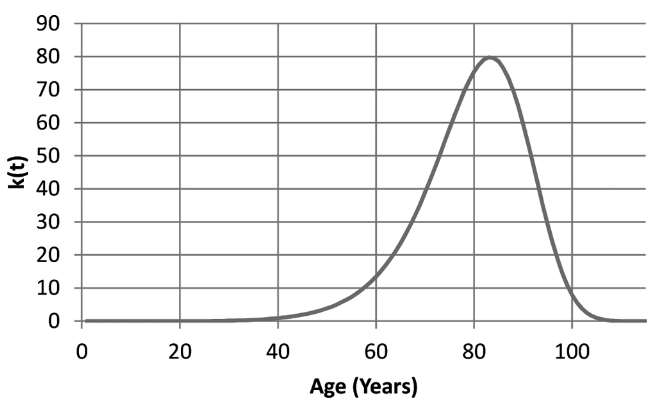

Figure 8. Estimation of $k(t)$ for the Skiadas model (10), ($c = 4$).

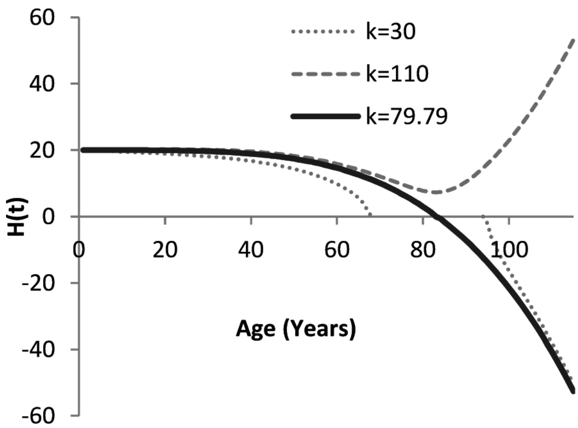

Figure 9. Reconstruction of $H(t)$ for the Skiadas model (10), ($c = 4$).

function (Pdf) by fitting a particular model to data sets, find the drift without estimating the diffusion coefficient or the infinitesimal variance σ^2, and second to use the drift to construct the stochastic model and then to produce sufficiently large number of stochastic paths crossing the barrier and producing the simulated probability density function (Spdf). The selection of the diffusion coefficient (not possible when fit a model to data) is done by making several selections as to minimize the error between SPDF and the PDF for a large number of stochastic simulations.

In this article, we use the facilities of the Excel program mainly because it is easy to apply by a large number of researchers, students or any interested person. The needed generators for stochastic simulations are included into the program thus easily producing a Wiener-type process and then estimating the stochastic realizations.

The first step is to check the hitting time simulation procedure by using a well-known Pdf, the so-called Inverse Gaussian given by the formula

$$g(t) = \frac{k}{\sqrt{t^3}} e^{-\frac{(H_t)^2}{2t}},$$

where H_t is provided in (10)

$$H_t = l - bt.$$

The stochastic paths are given by

$$S_t = H_t + \sigma \int_0^t dW_s. \tag{20}$$

A number of 308,655 stochastic simulations (Fig. 10A) for the simple model with $l = 20$, $\sigma = 1.2$, and $b = 0.6$ gave sufficiently good results (Fig. 10B). The barrier is set at zero. The Sum of Square Errors is negligible ($1.23*10^{-11}$). The mean value H_t is linear (see the straight line in Fig. 10A) as it was expected.

The next step is to apply the first exit time simulations method to the death and population data of a country. Here we use the data for the U.S. provided by the

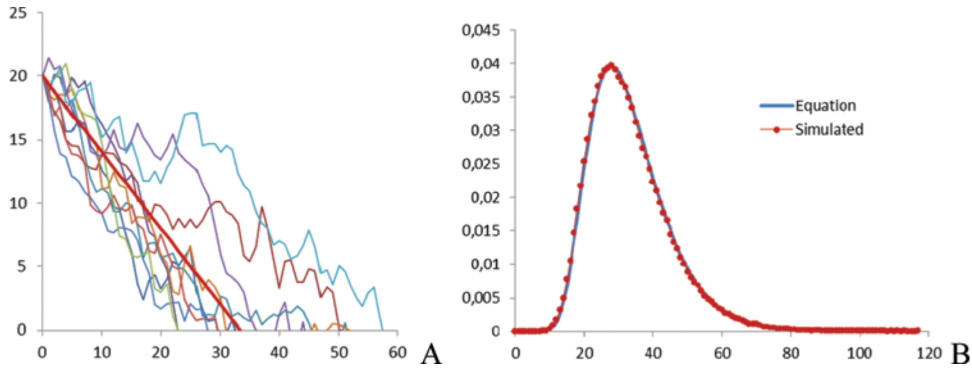

Figure 10. A. Stochastic simulations. B. The Pdf (equation) and Spdf (simulation).

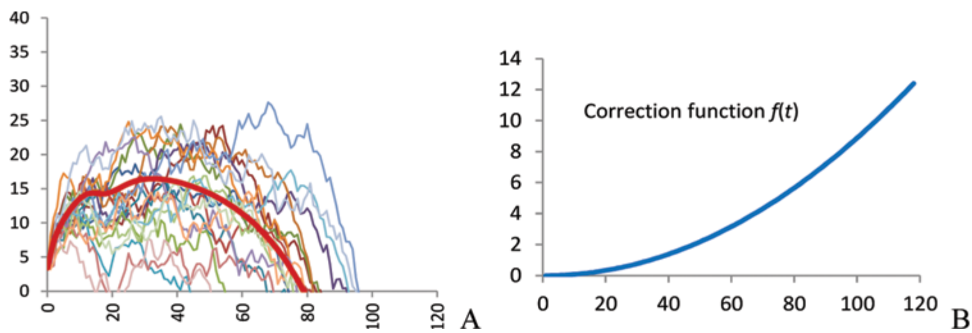

Figure 11. A. Stochastic simulations; B. The correction function f_t.

Human Mortality Database (HMD) already explored earlier. The mean value H_t is a smooth function presented by the heavy line in Fig. 11A where several stochastic paths for $\sigma = 1.2$ are illustrated. The health state function H_t is corrected according to the formula (9b). The parameters are given in Table 1. The form of this correction term is illustrated in Fig. 11B and the estimated and corrected values of H_t for the U.S. in 2000 are presented in Fig. 12A. The estimated H_t is based on the form (16). After 308,655 simulations the resulting Spdf is very close to the Pdf as it is illustrated in Fig. 12B.

Table 1

Life Expectancy at Birth and Correction Parameters for USA Males

Country/Time Period	Life Expectancy at Birth			Correction Parameters		Diffusion Coefficient
USA Males	Fit	SIM	HMD	a	β	σ
1935	58.61	58.89	58.96	0.0009054	2	1.25
1960	66.69	66.69	66.63	0.0006854	2	1.20
2000	74.31	74.59	74.19	0.0006754	2	1.20

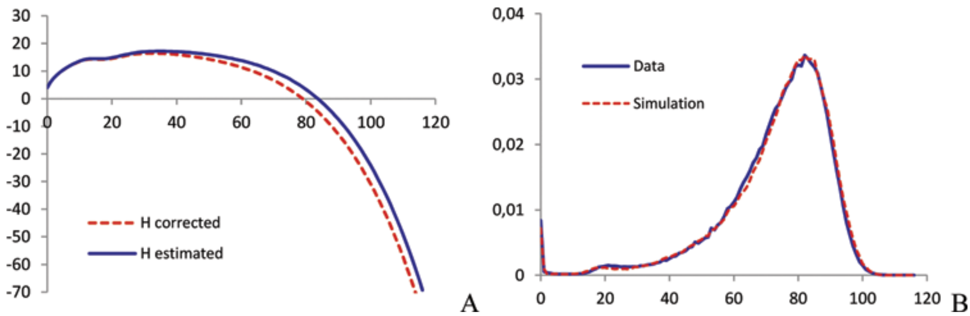

Figure 12. A. Estimated and corrected H_t; B. The Pdf (equation) and Spdf (simulation).

Table 1 summarizes the application results for males in the U.S. for the years 1935, 1960, and 2000. The Life Expectancy at Birth is estimated from the results of the fit curve (9) and from the simulation (SIM) and are compared with the estimates provided by the Human Mortality Database (HMD) indicating a very good, almost perfect approximation between these methods. The Sum of Square Errors is negligible, $3.25*10^{-6}$ for U.S. males for the year 1935 and also negligible for the years 1960 and 2000.

Another point when using the simulation methodology is the estimation of σ by doing very many estimates. However, for males in the U.S., the values for σ close to 1.2 are quite good selections thus simplifying and reducing the computation effort.

5. Conclusions

We have expanded and analyzed a general theory of health state of a population and various estimates useful in statistics and actuarial science. The aim was towards providing tools for measuring the health state of a population from life table data. By using this theory and the provided tools and the Excel programs the estimates and fitting curves are very easy. For information and downloads visit the website http://www.cmsim.net. We are now able to reconstruct the health state function from population and death data provided by the bureau of the census. The same tools can be introduced in many other fields.

Acknowledgments

The data used can be downloaded from the Human Mortality Database at: http://www.mortality.org or from the statistical yearbooks of the countries studied.

The models, methods, and fitting techniques and related programs can be downloaded from: http://www.cmsim.net

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