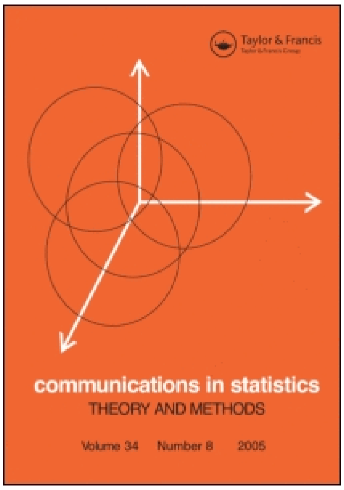

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Distributions, Models and Applications

Development, Simulation, and Application of First-Exit-Time Densities to Life Table Data

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In this article, we use the first-passage-time theory for a stochastic process to formulate a dynamic model expressing the human life table data. The model is derived analytically by using the corresponding Laplace transforms for the probability density function of the stochastic process and then the theory for the hitting time process is used to derive the probability density function for the first exit time. The tangent approximation to one-sided Brownian exit densities is used. The resulting probability density function is applied to mortality data. The stochastic simulation is using the Health State Function proposed with encouraging results.

Keywords First exit time; First-passage-time density; Health state function; Hitting time; Life table data; One-sided Brownian exit densities; Stochastic modeling; Stochastic simulation; Tangent approximation.

Mathematics Subject Classification Primary 60H10, 65C30, 82B31, 91B70; Secondary 60H30, 60H35, 62L20, 82C31.

1. Introduction

The construction of models for life table data based on the first-exit-time theory for stochastic processes has received considerable attention. The main problem when dealing with first-exit-time modeling when a process follows Brownian paths is that, in general, no closed form solutions can be obtained, except for a few very simple cases. The simplest such case is to consider Brownian stochastic paths crossing a barrier in the form of a straight line. In this case, the probability density function obtained has a very simple closed form, also known as the *Inverse Gaussian*.

Schrödinger (1915) and Smoluchowsky (1915) were the first to introduce the Inverse Gaussian distribution in connection to the first hitting time in a system

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where the particles are following a Brownian motion. Since then the Inverse Gaussian has been studied extensively by many authors. See the early studies by Tweedie (1945, 1946) and Wald (1947), and later by Levy (1965) and Shuster (1968). The work by Siegert (1951) is mainly influenced by the first articles by Schrödinger and Smoluchowski. The inverse Gaussian has been used as a model for the distribution of life table data (see Chhikara and Folks, 1977, 1989).

However, the Inverse Gaussian is a model with limited ability to express first-exit-time cases when the exit boundary is nonlinear. This is the case for several theoretical but also applied situations. Then the resulting forms of the probability density function (pdf) for nonlinear boundaries are not easily expressed in closed form, thus making difficult or impossible to use in real life applications. The situation changed with the use of computers in the past decades. The tendency has been in the direction of finding good approximations for some cases of first-exit-time densities from nonlinear boundaries. These approximations are based on the works of Jennen (1985), Lerche (1986), Jennen and Lerche (1981), Ferebee (1982, 1983), Daniels (1996), Sacerdote (1990), Durbin (1992), and others, and gave rise to several interesting theoretical and applied studies. Between these studies was the introduction of a first-exit-time probability density function to express the life table data of a population (Janssen and Skiadas, 1995). Several applications to the same field were presented (see Skiadas and Skiadas, 2007; Skiadas et al., 2007). The idea in the original article of 1995 was to assume that the health state function is a stochastic process and that the life status of a living organism is determined by a boundary at a minimum level for the health state.

The idea is simple and it can be modeled provided that some theoretical issues can be resolved. There are enough data to support real life applications and especially the life table data for a population provided by the Bureau of the Census of every country.

The original model proposed by Janssen and Skiadas (1995) was *too heavy* for the applications needed as many parameters where inherent. However, the model had the ability to capture infant mortality into the pdf proposed for a population. Simpler models with good forecasting and explanatory abilities where proposed in last few years.

In this article, an improved version is proposed, derived analytically, and applied to life table data from the United States. The transition pdf of the stochastic process is found by solving the Fokker–Planck equation of the process. Then the *tangent approximation* proposed by Jennen and Lerche (1981) is used to find a good approximation for the pdf of the first-exit-time of a stochastic process from a curved boundary.

2. Health State as a Stochastic Process

The health state of an individual can be modeled as a stochastic process with a mean "health state" function $H(t)$, satisfying the stochastic differential equation

$$dS(t) = \mu(t)dt + \sigma(t)dW(t), \tag{1}$$

where

$$\mu(t) = \frac{\partial H}{\partial t}$$

and W is the familiar Wiener process. Throughout this article, we will assume time evolution as starting from $t = 0$. The fundamental assumption is that an individual dies when their health state $S(t)$ reaches a certain threshold value, namely, 0.

A convenient form for the health state function is $H(t) = c - (\ell t)^b$, where $c = H_0$ is the initial state of the individual, and we will be examining this particular case later on. For now, note that under the simplifying assumption that σ is constant, the above stochastic differential equation is directly integrable and has the solution

$$S(t) = H(t) + \sigma W(t). \tag{2}$$

The stochastic process S is associated with the following transition probability function P:

$$P(x, t; x_0, t_0) = P[S(t) \leq x \mid S(t_0) = x_0] \tag{3}$$

with $t_0 < t$ and $x, x_0 > 0$.

Under the assumption that the stochastic process $S(t)$ is continuous the transition probability density function, usually expressed by a Chapman–Kolmogorov equation reduces to the following forward equation for the transition densities, also known as the Fokker–Planck equation:

$$\frac{\partial p(S(t), t)}{\partial t} = -\mu(t)\frac{\partial\big(p(S(t), t)\big)}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2\big(p(S(t), t)\big)}{\partial S^2}. \tag{4}$$

Under appropriate boundary conditions this Fokker–Planck equation can be solved explicitly. We choose the following boundary conditions:

$$\begin{aligned} p(S(t), 0; S_0, 0) &= \delta(S(t) - S_0) \\ \frac{\partial p\big(S(t), t; S_0, t\big)}{\partial S(t)} &\to 0 \quad \text{as } S(t) \to \pm\infty. \end{aligned} \tag{5}$$

Equation (3) under conditions (4) can be solved by introducing the characteristic function $\phi(\xi, t)$ via the transformation:

$$\phi(\xi, t) = \int_{-\infty}^{\infty} p(S, t; S_0, 0)e^{i\xi S}dS. \tag{6}$$

Using (4) and integrating by parts we have:

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= \int_{-\infty}^{\infty}\frac{\partial p}{\partial t}e^{i\xi S}dS \\ &= -\mu(t)\int_{-\infty}^{\infty}\frac{\partial p}{\partial S}e^{i\xi S}dS + \frac{1}{2}\sigma^2\int_{-\infty}^{\infty}\frac{\partial^2 p}{\partial S^2}e^{i\xi S}dS \\ &= i\xi\mu(t)\phi - \frac{1}{2}\sigma^2\xi^2\phi. \end{aligned} \tag{7}$$

This first-order differential equation can be easily solved, and along with the initial conditions from (5), which take the form

$$\phi(\xi, 0) = e^{i\xi S_0} = e^{i\xi H_0}, \tag{8}$$

we arrive at the formula

$$\phi = \exp\left(i\xi H(t) - \frac{1}{2}\sigma^2\xi^2 t\right). \tag{9}$$

As can be seen by a simple direct integration (see also Feller, 1986), this ϕ is the characteristic function of a Gaussian with mean $H(t)$ and variance $\sigma^2 t$.

Therefore, $p(S, t; S_0, 0)$ will have the form:

$$p(S, t; S_0, 0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{\left(S - H(t)\right)^2}{2\sigma^2 t}\right). \tag{10}$$

For the special case where H is a decreasing power function $H(t) = S_0 - (\ell t)^b$, the last formula becomes:

$$p(S, t; S_0, 0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{\left(S - S_0 + (\ell t)^b\right)^2}{2\sigma^2 t}\right). \tag{11}$$

The special case $b = 1$, $l = \mu$, corresponding to a constant drift, is well known:

$$p(S, t; S_0, 0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(S - S_0 + \mu t)^2}{2\sigma^2 t}\right). \tag{12}$$

3. The Hitting Time Distribution and Life Table Data

We are interested in particular at the so-called first-exit-time or hitting-time distribution, namely the time at which the process drops below a particular barrier 0, corresponding to the death of the individual. For the special case of the stochastic process described with (12), the hitting-time distribution (with $\mu = 0$ or $\mu =$ constant) was studied by Schrödinger (1915) and Smoluchowsky (1915), and later by Siegert (1951), and has the form:

$$\begin{aligned} g(0, t; S_0, 0) &= \frac{|S_0|}{t} p(0, t; S_0, 0) \\ &= \frac{|S_0|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{(S_0 - \mu t)^2}{2\sigma^2 t}\right). \end{aligned} \tag{13}$$

In the more general case of (11), the first passage passage time density function, $g(t) = g(0, t; S_0, 0)$, is found by using the *tangent approximation* (see Jennen, 1985) of the first-exit density, and takes the form:

$$\begin{aligned} g(0, t; S_0, 0) &= \frac{|S_0 + (b-1)(\ell t)^b|}{t} p(0, t; S_0, 0) \\ &= \frac{|S_0 + (b-1)(\ell t)^b|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{\left(S_0 - (\ell t)^b\right)^2}{2\sigma^2 t}\right) \end{aligned} \tag{14}$$

More generally, the hitting time distribution for the case of the process (2) is approximated by:

$$g(0, t; S_0, 0) = \frac{|H(t) - tH'(t)|}{t} p(0, t; S_0, 0)$$
$$= \frac{|H(t) - tH'(t)|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{H(t)^2}{2\sigma^2 t}\right) \tag{15}$$

Based on this distribution, we propose that life table data be modeled with the function:

$$g(t) = k \frac{\left|c + (b-1)(\ell t)^b\right|}{(\ell t)^{3/2}} \exp\left(-\frac{\left(c - (\ell t)^b\right)^2}{2t}\right) \tag{16}$$

The conditions presented in Jennen (1985) are quite strict to make them not theoretically applicable in most cases. However, the simulations performed in this article show that (15) is an adequate approximation to the first-exit-time density. We perform in this article simulations for various parameter values, including the parameters corresponding to data fits from the United States life table data for the year 2000.

4. Simulations

Our basic setup is as follows. We consider the barrier to be at 0, and the process to start from a positive health state $S_0 = c$. We then follow the stochastic process (2). Since the barrier is set at 0, a rescaling of (2) will not alter the hitting-time distribution, therefore we may assume without loss of generality that the process has been so rescaled that $\sigma = 1$. The health state function $H(t)$ has the form $H(t) = c - (\ell t)^b$. The overall form of the stochastic process therefore becomes:

$$S(t) = c - (\ell t)^b + W(t)$$

The time of death is determined as the time when $S(t)$ drops below the barrier 0. This process then gives rise to the hitting time distribution (14), up to approximation errors. We determine parameters by fitting the model to the U.S. data. Values have been scaled so that they add up to 1,000, and deaths that occur at ages less than 1 were removed. The parameters for the two cases are:

	Females	Males
b	5	4.6
k	1.077	1.2
ℓ	0.02	0.022
c	13.35	11.32

The simulation was performed as follows. The Wiener process is simulated for time intervals of step 0.1, and then added to the health state function. Results are aggregated to integer points. Figure 1 show the results of the simulation. The solid line corresponds to the approximation to the theoretical first-exit-time distribution

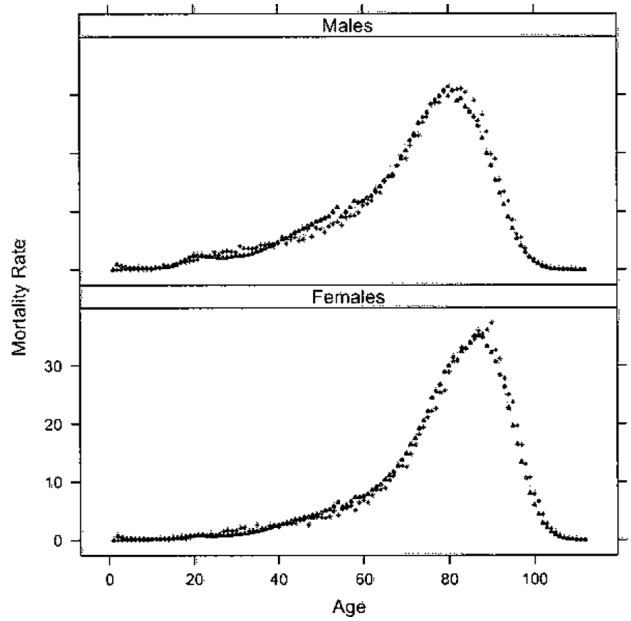


Figure 1. Mortality data from the U.S. for 2000 (crosses), and corresponding simulated data (triangles). The solid lines correspond to the approximation to the theoretical first-exit-time distribution.

g, while the triangle and cross marks correspond to the simulated and actual data, respectively.

In Fig. 2, the results for a number of simulations with different parameter values are shown. In all cases $\sigma = 1$, and ℓ is selected so that the health state function

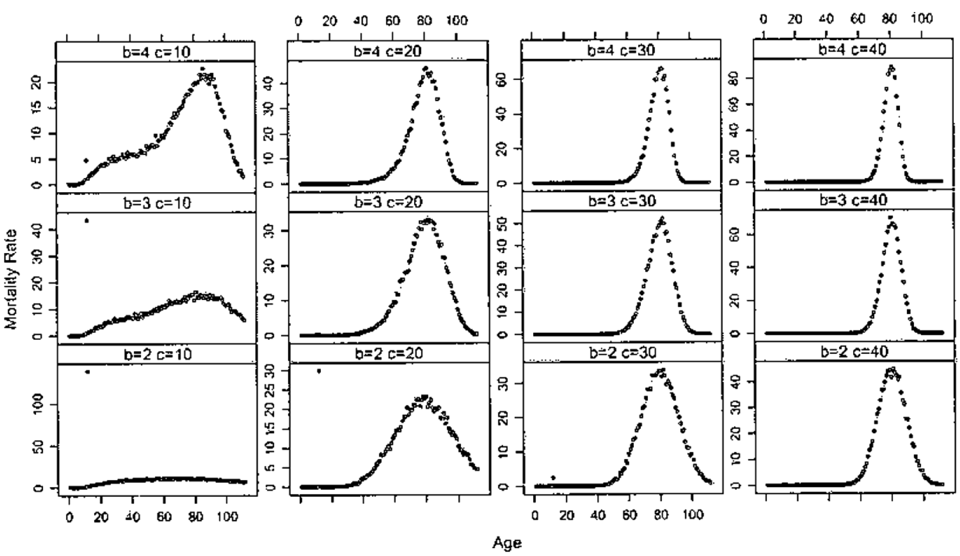
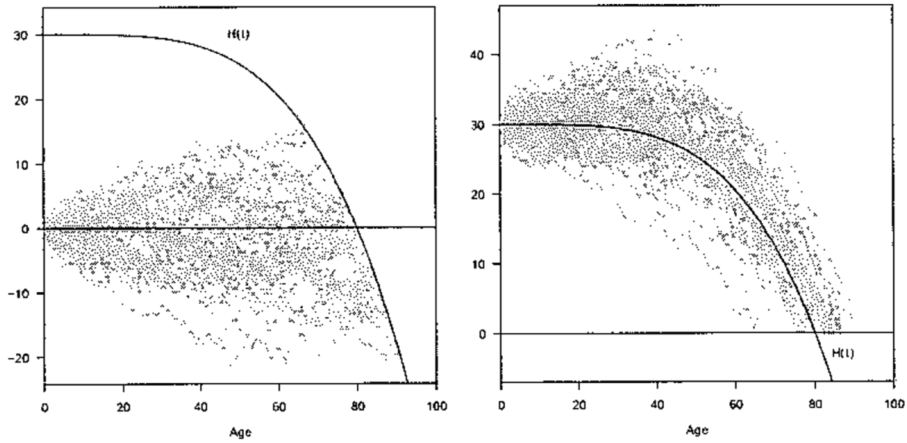


Figure 2. Life table simulation results. The solid line corresponds to the theoretical curve, while the points are the simulated data.



(a) (b)

Figure 3. Illustration of the stochastic paths in life table data simulation. (a) Random walks meeting the health state function $H(t)$. Time of contact is the first-exit-time. (b) Stochastic paths around the health state function. The first-exit-time corresponds to crossing the x-axis.

becomes 0 at age 80. The parameter b takes the values 2, 3, and 4, while c takes values 10, 20, 30, 40. It can be clearly seen that the simulations match the theoretical distribution provided the starting health state c is sufficiently far from the threshold barrier 0, relative to σ. Note however the very unusual behaviour around age 15 in some of the simulations. This warrants further investigation.

For completeness, we demonstrate in Fig. 3 a simulation of the random walks. The simulation can be performed in two obviously equivalent ways.

1. Follow a Wiener process until it crosses the health state function H, i.e., check for the condition $W(t) \geq H(t)$. This is shown in Fig. 3(a), where the paths correspond to the standard Wiener process, and they continue until they cross the decreasing health state function H.
2. Follow the process $H(t) + W(t)$, and observe where it crosses the x-axis, i.e., check for the condition $H(t) + W(t) \leq 0$. This is shown in Fig. 3(b), where the various stochastic paths move around the health state function, until they hit the x-axis barrier. The equivalence is clear because of the symmetry of the Wiener process.

5. Conclusions

The approximation to the first-exit-time distribution proposed by Jennen (1985) and elsewhere appear in these simulations to perform quite adequately, and provide models that fit the real life table data very satisfactorily. The modeling Eq. (16), for the parameters resembling those from life table data, was seen in the simulated data to describe the pdf of the first-exit-time very well.

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