COMPARING THE GOMPERTZ TYPE MODELS WITH A FIRST PASSAGE TIME DENSITY MODEL

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ABSTRACT. In this paper we derive and analyse Gompertz type probability density functions and compare these functions to a first passage time density function. The resulting Gompertz type pdfs are mirror images of each other, each skewed in a specific direction, whereas the first passage type model gives functions with both left and right skeweness depending on parameter values. We apply these pdfs to the Life Table Data for females in the United States, 2004, and to the medfly data provided in Carey, 1992. Our application shows that the mortality data in the two cases have opposite skewness. The results support that the underlying mortality mechanism is qualitatively different in the cases.
Keywords: Gompertz, Mirror Gompertz, Inverse Gompertz, Dynamic model, Probability density function, Life table data, Life table data, Weibull

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1. INTRODUCTION

There is extensive bibliography concerning the famous Gompertz model and its applications to Life Table Data. The questions raised by Robine,1993 in response to Carey, 1992 suggest comparing the medfly life span modeling data to the human life span data.

As Carey, 1992 state, the experiment was mainly designed to explain longevity. However, the majority of the data collected, for more than 1.200.000 medflies, are not appropriate to support longevity studies, but instead are mainly to be used for explaining the mortality law for medflies. It is worth noting that the medfly data was easily verified to follow the Gompertz probability density function. Later on, Weitz, 2001 suggested a first passage probability density function to model the medfly data. The proposed model type was first derived independently by Schrödinger, 1915 and Smoluchowsky, 1915, and in a more recent form by Siegert, 1951, and it is known as the *inverse Gaussian distribution*. In recent years we can find various applications of this model to lifetime data analysis and reliability, and other fields.

A more general *first passage density function* was proposed by Janssen and Skiadas, 1995 to model the human life table data. This model was applied to the data provided from the life table records of Belgium and France, and it gave a very good fitting. However, it was a model difficult to work with, as it contained many parameters. A simpler first exit time density function was proposed by Skiadas et

[1]Draft version, 2008. The final version is included in the book: *Advances in Data Analysis*, C. H. Skiadas (Ed) Springer/Birkhauser, 2010, pp. 203-209.

al, 2007 and Skiadas and Skiadas, 2007, with applications to life table data from Greece. The special case of a quadratic health state function was discussed in Skiadas et al, 2007. The first exit (or *hitting*) time density has the form:

$$g_{\mathrm{DM}}(t) = c(kt)^{-\frac{3}{2}} e^{-\frac{\left(\ell-(kt)^b\right)^2}{2t}}, \tag{1}$$

where c, ℓ and k are parameters, and b is a constant mainly related with the skewness of the probability density function. The (simpler) inverse Gaussian distribution model corresponds to the case where $b = 1$.

The model (1.1) was tested using the life table data from Greece (1992–2000), and it showed very good fitting. More importantly, the term

$$H(t) = \ell - (kt)^b, \tag{2}$$

called the *health state function* in Skiadas and Skiadas, 2007, provides useful information on the mortality data applied; Namely, it describes the perceived average health state of an individual for a given age. Equation (1.2) indicates a health state that decays over time, slowly at first and faster as the age of individuals approaches its natural limits.

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2. The Gompertz type models

The Gompertz model was first proposed by Gompertz, 1825, and a thorough analysis of it can be found in Winsor, 1932. A derivation based on systems theory is done by Skiadas and Skiadas, 2008. In differential equation form, the model has the equation

$$(\ln x)' = -b \ln x, \tag{3}$$

or equivalently

$$\dot{x} = -bx \ln x, \tag{4}$$

where x is a function of time t, and b is a positive parameter expressing the rate of growth of the system. Without loss of generality the function x can be assumed bounded ($0 < x \leq 1$, with $x = 1$ corresponding to the entire population), so that $\dot{x}$ is the probability density function of the growth process. Direct integration of eq. (1.1) gives as solution the *Gompertz function*:

$$x = e^{\ln(x_0)e^{-bt}} \tag{5}$$

The probability density function of the Gompertz model is then given by

$$g(t) = \dot{x} = -b \ln(x_0) e^{-bt} e^{\ln(x_0)e^{-bt}}. \tag{6}$$

An interesting variant of the Gompertz function arises when we replace x by $1 - x$ in the right side of the Gompertz differential equation, resulting in a mirror image of the Gompertz model (see Skiadas and Skiadas, 2008):

$$\dot{x} = -b(1-x)\ln(1-x) \tag{7}$$

Direct integration gives as solution the *Mirror Gompertz function*:

$$x = 1 - e^{\ln(1-x_0)e^{bt}} \tag{8}$$

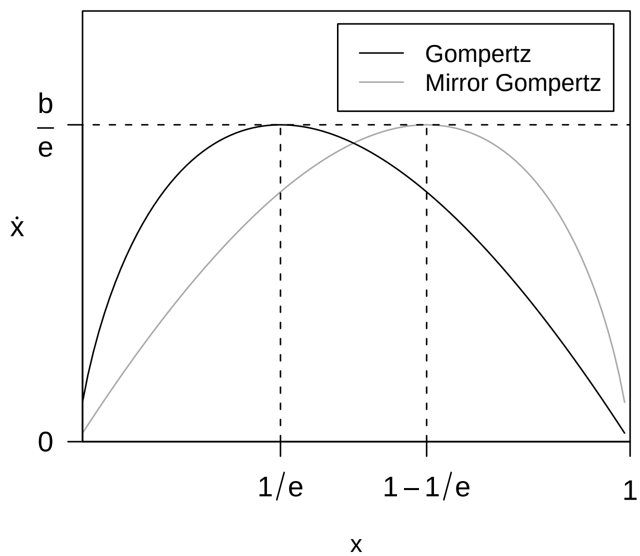

FIGURE 1. The two Gompertz type models

The probability density function of the Mirror Gompertz model is then given by

$$g_{\mathrm{MGM}}(t) = \dot{x} = -b\ln(1 - x_0)e^{bt}e^{\ln(x_0)e^{bt}}. \tag{9}$$

This model arises by considering the *relative decay* of the system, instead of the *relative growth*. It has skewness opposite that of the Gompertz model, as now the maximum growth rate is achieved when

$$x = 1 - \frac{1}{e}$$

instead of the Gompertz model when $x = \frac{1}{e}$. A comparison of the two models is given in Figure 1.1, with the Mirror Gompertz model appearing in gray. Both models are referred to in the literature as "the Gompertz model", with different disciplines preferring one model over the other. The second variant is favoured in the actuarial sciences, as it is more intimately related to mortality.

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3. APPLICATION TO LIFE TABLE AND THE CAREY MEDFLY DATA

The Carey data are provided in his famous Science paper, Carey, 1992. Since then, several papers with further analyses and applications have appeared. The data used in this study are selected from a laboratory experiment where the life span of 1.203.646 medflies was measured.

Weitz, 2001 used the medfly data to test the inverse Gaussian distribution as a model resulting from the *first exit time theory.* The fitting of this model was quite good. In the present study we fit the more general model given by equation (1.1) to the data, and we test whether the exponent b diverges from unity or not ($b = 1$ being exactly the case studied in Weitz, 2001). In the same study we will test the argument of Robine, 1993 regarding a comparative study of the medfly life span and the human life span. The United States 2004 life table data for females was used for the comparative study. Four models are tested. The equations used for the data fit are:

$$\text{(Dynamic Model)} \qquad g_{\mathrm{DM}}(t) = c(kt)^{-3/2}e^{-\frac{(\ell-(kt)^b)^2}{2t}}$$

$$\text{(Gompertz Model)} \qquad g_{\mathrm{G}}(t) = ce^{-kt}e^{-\ell e^{-kt}}$$

$$\text{(Mirror Gompertz)} \qquad g_{\mathrm{MG}}(t) = ce^{kt}e^{-\ell e^{kt}}$$

$$\text{(Weibull)} \qquad g_{\mathrm{W}}(t) = c(kt)^{\ell-1}e^{-(kt)^{\ell}}$$

For the Dynamic Model, the parameter b was 1.4 for the Medfly data and $b = 5.76$ for the life table data for females 2004 in USA.

The results are summarized in Table 1 for USA 2004 females. As can be seen

	USA Data Females 2004 Fit			
Model	c	k	ℓ	MSE (10^{-4})
Dynamic Model	0.0724	0.01875	17.318	5.85
Mirror Gompertz	0.0000221	0.09776	0.000241	13.71
Weibull	0.09095	0.01167	8.64	20.17

Table 1. Fit comparison for USA 2004, Females.

from Table 1.1, the best fit is done by the Dynamic model with parameter $b = 5.76$ and a Mean Squared Error MSE = 5.85, followed by the Mirror Gompertz with a MSE = 13.71 and finally the Weibull model with MSE = 20.17.

For the Carey medfly data the fitting results are summarized in Table 1.2. The

	Carey Medfly Data Fit			
Model	c	k	ℓ	MSE (10^{-4})
Dynamic Model	0.396	0.2085	8.193	11.56
Gompertz	1.31335	0.13715	9.61624	9.68
Weibull	0.1197	0.046596	2.656	16.91

Table 2. Fit comparison for Carey Medfly Data.

best model fit is now given by the Gompertz model, followed closely by the Dynamic model with $b = 1.4$. The Weibull model performs much worse. Regarding the Weitz, 2001 application with $b = 1$, the resulting fitting error is MSE = 13.59, that is, higher than both the Gompertz and the Dynamic model. The estimated parameters for the Weitz and Fraser application are $c = 1.32336$, $k = 0.47726$ and $\ell = 10.2200$ by following the method of nonlinear least squares estimation applied here. The Weibull model, in both cases presented here, did not give results as good as the Gompertz and the Dynamic Models.

Figure 1.2 illustrates the fit comparison between the Medfly data and the USA 2004 females data by using the Gompertz and Mirror Gompertz models respectively (gray lines), as well as the proposed Dynamic Model (black line). The time scale

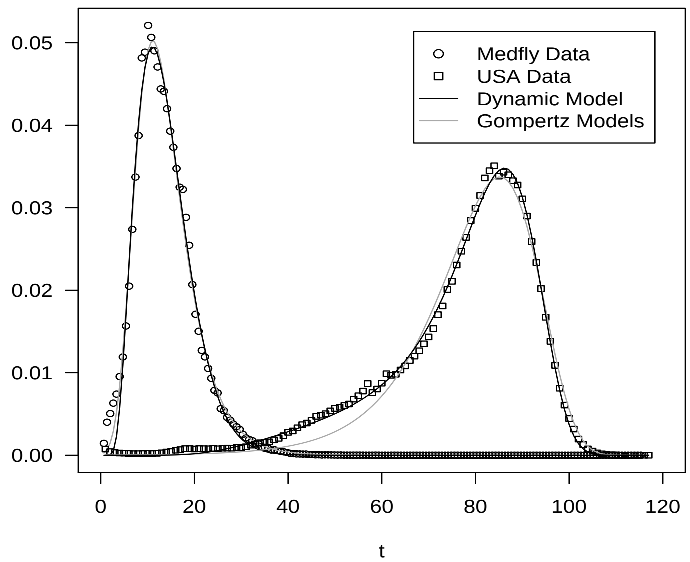

FIGURE 2. Gompertz, Mirror Gompertz and Dynamic models applied to the Medfly and USA 2004 Female data

was re-scaled according to the method proposed by Robine, 1993. It is clear that the Medfly data are very well presented by the Gompertz model, whereas the human mortality data are well expressed by the Mirror Gompertz.

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4. Remarks

According to a theory based on the *tangent approximation* (see Jennen, 1985 and Jennen, 1981) the hitting time distribution for the case of the Health State Process $H(t)$ presented above, is approximated by:

$$\begin{aligned} g(t) &= \frac{|H(t) - tH'(t)|}{t} p(t) \\ &= \frac{|H(t) - tH'(t)|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{H(t)^2}{2\sigma^2 t}\right) \end{aligned} \tag{10}$$

Where $p(t)$ is the transition probability density function. Based on this distribution, we propose that the life table data be modelled with the function:

$$g(t) = \frac{\left|\ell + (b-1)(kt)^b\right|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{\left(\ell - (kt)^b\right)^2}{2\sigma^2 t}\right) \tag{11}$$

This function offers fits very similar to those of the model suggested here, and has better simulation properties, that will be examined in upcoming papers.

5. Conclusion

In this paper we presented a comparative study including two Gompertz type models and a Dynamic model. The Weibull model was also tested in the applications. The application of the Gompertz and Mirror Gompertz, and of the Dynamic model, to explain the behavior of mortality data was very promising, both from a fitting point of view, but also from an explanatory point of view.

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